The exceptional zero conjecture for symmetric powers of CM modular forms: the ordinary case

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Abstract

We prove the exceptional zero conjecture for the symmetric powers of CM cuspidal eigenforms at ordinary primes. In other words, we determine the trivial zeroes of the associated p-adic L-functions, compute the L-invariants, and show that they agree with Greenberg's L-invariants.

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Introduction

The exceptional zero conjecture concerns the zeroes of p-adic L-functions occurring at critical integers for a *trivial reason* (namely, the vanishing of the interpolation factor). Given a (p-ordinary) motive M, there is conjecturally a p-adic function $L_p(s, M, \chi)$ (the *analytic p-adic L-function of M*) interpolating the twisted values of the (usual) archimedean L-function $L(s, M, \chi)$ at its critical integers (in the sense of Deligne's article [D79]). This interpolation takes the form

$$L_p(a, M, \chi) = \mathcal{E}(a, M, \chi)\frac{L(a, M, \chi^{-1})}{\Omega_{a,M,\chi}} \tag{1}$$

where $\mathcal{E}(a, M, \chi)$ is a fairly simple arithmetic fudge factor called the *interpolation factor*, $\Omega_{a,M,\chi}$ is some period (present to obtain algebraicity of values), and a varies over the critical integers of M. For certain choices of a and χ (and M), it can happen that $\mathcal{E}(a, M, \chi)$ vanishes, causing a trivial vanishing of the p-adic L-function. Such an occurrence is called a *trivial zero* (or an *exceptional zero*) of $L_p(s, M, \chi)$. Since the p-adic interpolation of L-values is a fruitful approach to understanding their arithmetic, it is worth attempting to correct the loss of information that occurs at a trivial zero. This is where the exceptional zero conjecture comes in. Supposing $\mathcal{E}(a_0, M, \chi_0) = 0$ and $L(a_0, M, \chi_0) \neq 0$, let $\mathcal{E}^+(a_0, M, \chi_0)$ be $\mathcal{E}(a_0, M, \chi_0)$ with its vanishing factors removed.[1] The exceptional zero conjecture for (a_0, M, χ_0) can be then be stated in two parts.

Conjecture 0.1 (Exceptional zero conjecture, vaguely). [2]

(i) *There is an "analytic L-invariant"* $\mathcal{L}_p^{\mathrm{an}}(a_0, M, \chi_0) \in \mathbf{C}_p^\times$ *such that*

$$\lim_{s \to a_0} \frac{L_p(s, M, \chi_0)}{(s - a_0)^e} = \mathcal{L}_p^{\mathrm{an}}(a_0, M, \chi_0)\mathcal{E}^+(a_0, M, \chi_0)\frac{L(a_0, M, \chi_0^{-1})}{\Omega_{a_0,M,\chi_0}},$$

where e is the "order of vanishing of $\mathcal{E}$".

(ii) *There is an "arithmetic L-invariant"* $\mathcal{L}_p^{\mathrm{arith}}(a_0, M, \chi_0) \in \mathbf{C}_p$, *defined in terms of the arithmetic of M, such that*
$$\mathcal{L}_p^{\mathrm{an}}(a_0, M, \chi_0) = \mathcal{L}_p^{\mathrm{arith}}(a_0, M, \chi_0).$$

Part (i) conjectures a way of recuperating an interpolation property, whereas part (ii) is a supplement to an *Iwasawa–Greenberg Main Conjecture* in the setting of a trivial zero, namely it gives an arithmetic meaning to an analytically defined L-invariant.

In Theorems A and B below, we prove this conjecture for motives arising from symmetric powers of p-ordinary newforms *with complex multiplication*. More specifically, the adjoint motive of a newform, which has important applications to its deformation theory, is the symmetric square motive twisted by the inverse of the determinant and we consider appropriate determinant twists of higher symmetric powers. The arithmetic L-invariant we use is that suggested by Greenberg in [G94]. We will denote it $\mathcal{L}_p^{\mathrm{Gr}}(a_0, M, \chi_0)$. In the CM case, the symmetric powers decompose into a sum of Tate twists of modular forms and sometimes a Dirichlet character, with the trivial zero coming from the latter factor. We prove our theorems by reducing them to these already known cases. In particular, the determination of the L-invariant is reduced to the results for Dirichlet characters of Ferrero–Greenberg in [FeG78] and Gross and Koblitz in [GrKo79] and [Gr81, §4]. In the specific case of the symmetric square of a CM elliptic curve, part of our results were obtained by Dabrowski and Delbourgo in [DabDel97, Theorem 3.6.1]. Other related results in the literature on symmetric powers of modular forms mostly concern the determination of Greenberg's L-invariant. The case of a CM elliptic curve was done in [G94]. Greenberg also covers the case of elliptic curves with split, multiplicative reduction at p in that paper. Generalizations of this latter result

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[1]The interpolation factor is a product of "Euler-like" factors, some of which might vanish.

[2] We say *vaguely* here not only because we have left several terms undefined, but mostly since our notation $L_p(s, M, \chi)$ is more evocative than accurate. Really there are p branches of the p-adic L-function and the limit should be taken along the branch where the trivial zero occurs. For the precise formula we prove, see Theorem A below.

to higher weights and to Hilbert modular forms have been obtained by Hida (see, for example, [Hi07]). However, the most difficult case to consider is the non-CM case when p does not divide the level of the modular form f. A formula for Greenberg's L-invariant was obtained by Hida in the symmetric square case in [Hi04] under some technical assumptions (see also chapter 2 of the author's Ph.D. thesis [H-PhD] for a slightly different approach). In [PR98], Perrin-Riou has a conditional result on the analytic L-invariant of the symmetric square of an elliptic curve. The author has addressed the next highest symmetric power L-invariant (i.e. the sixth power) in [H10]. The author's current strategy for dealing with higher symmetric powers requires that f not be CM, so it is important to deal with the (simpler) CM case separately.

Based on what happens for his L-invariant of elliptic curves with CM or with split, multiplicative reduction at p, Greenberg has raised the question of whether or not the L-invariant is independent of the symmetric power n that you take ([G94, p. 170]). Our results show that independence holds for CM newforms.

Finally, we wish to mention two ways to extend these results. The first, is to consider the non-ordinary situation. In this case, the Dirichlet character occurring in the decomposition of the symmetric power does not contribute a trivial zero. In fact, the trivial zeroes come from the (odd weight) modular forms in the decomposition. This should lead to trivial zeroes of higher order and the determination of the L-invariant should reduce to the recent work of Denis Benois in [Be11]. The author plans to investigate this case. In another direction, one may pass from $\mathbf{Q}$ to a totally real field F and consider the L-invariants of symmetric powers of the Hilbert modular forms that arise from the induction of a Hecke character of a CM extension of F. Let us mention that the generalization of Gross' result has recently been studied by Dasgupta–Darmon–Pollack in [DDP11].

Notation and conventions

Throughout, we fix a rational prime $p \geq 3$, algebraic closures $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ and $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, and embeddings $\iota_\infty : \overline{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. The choice of ι_p determines an (arithmetic) Frobenius element $\mathrm{Frob}_p \in G_{\mathbf{Q}}$ (up to inertia). Similarly, ι_∞ gives a complex conjugation $\mathrm{Frob}_\infty \in G_{\mathbf{Q}}$. Denote by $\chi_p : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ the p-adic cyclotomic character. Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ and let $\log_p : \mathbf{C}_p^\times \to \mathbf{C}_p$ denote the Iwasawa branch of the p-adic logarithm (i.e. such that $\log_p(p) = 0$).

Dirichlet characters and p-adic L-functions

A Dirichlet character will be a character $\chi : (\mathbf{Z}/\mathfrak{f}\mathbf{Z})^\times \longrightarrow \mathbf{C}^\times$. Its conductor will be denoted $\mathfrak{f}_\chi$ and its Gauss sum $\tau(\chi)$. We will identify χ with a Galois character, also denoted χ, using the normalization $\chi(\mathrm{Frob}_p) = \chi(p)$ for $p \nmid \mathfrak{f}$. A Dirichlet character is even or odd according to whether $\chi(-1) = \chi(\mathrm{Frob}_\infty)$ is $+1$ or -1. The L-function of the Dirichlet character χ is defined for $\mathrm{Re}(s) \gg 0$

by

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

Let $\Gamma^\times := \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ be the Galois group over $\mathbf{Q}$ of the field of p-power roots of unity. We will consider our p-adic L-functions as coming from p-adic measures (or pseudo-measures) on $\Gamma^\times$ (see, for example, [Cz00] for these notions). If μ is such a (pseudo)-measure, it can be evaluated at elements χ in (some subset of) $\mathfrak{X}^\times := \mathrm{Hom}_{\mathrm{cts}}(\Gamma^\times, \mathbf{C}_p^\times)$, the continuous group homomorphisms valued in $\mathbf{C}_p^\times$. The cyclotomic character gives an isomorphism $\chi_p : \Gamma^\times \xrightarrow{\sim} \mathbf{Z}_p^\times$ giving two natural characters $\omega, \langle \cdot \rangle \in \mathfrak{X}^\times$ obtained by composing the cyclotomic character with the natural projections onto the first and second factors of $\mathbf{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbf{Z}_p)$. The finite order characters in $\mathfrak{X}^\times$ are just those arising from p-power conductor Dirichlet characters (using ι_∞ and ι_p). Each such character can be written uniquely as $\chi = \omega^a \varphi$ where $0 \leq a < p - 1$ and φ is a character of $1 + p\mathbf{Z}_p$ (and hence necessarily even). If M is some "object" and μ_M is a (pseudo-)measure attached to it, we will denote by $L_{p,i}(s, M)$ the p-adic function on a subset of $\mathbf{Z}_p$ given by

$$L_{p,i}(s, M) := \int_{\Gamma^\times} \omega^i \langle \cdot \rangle^s d\mu_M$$

and call it the ith branch of the p-adic L-function of M. We use the shorthand

$$L_p(s, M, \chi) := \int_{\Gamma^\times} \chi \chi_p^s d\mu_M$$

where χ is assumed to be of finite order and which makes sense for $s \in \mathbf{Z}$.

Hecke characters and CM modular forms

We will be studying holomorphic cusp forms of integral weight $k \geq 2$ on the upper-half plane. We refer to [Ri77] for the notion of newform with Nebentypus and to sections 3 and 4 of that article for the notion of, and basic facts concerning, CM modular forms. In particular, the term newform will imply a normalized cuspidal eigenform for the Hecke algebra generated by all Hecke operators T_n. Our conventions regarding Hecke characters also follow *loc. cit.* Finally, if

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$$

is the q-expansion of the newform f, then its L-function is defined for $\mathrm{Re}(s) \gg 0$ by

$$L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s}.$$

1 Symmetric powers of ordinary CM modular forms

Let f be a p-ordinary[3] newform of weight $k \geq 2$, of Γ_1-level prime to p with Nebentypus character ψ. Let $\mathbf{Q}(f) \subseteq \mathbf{C}$ be the field generated by the Fourier coefficients of f. Let K be the completion of $\iota_p \iota_\infty^{-1} \mathbf{Q}(f)$ in $\overline{\mathbf{Q}}_p$ and let $\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}(V_f)$ be the contragredient of the p-adic Galois representation (occurring in étale cohomology) attached to f by Deligne ([D71]) on the two-dimensional vector space V_f over K. Its determinant is $\psi \chi_p^{k-1}$. Wiles ([W88, Theorem 2.1.4]) shows that, for f as above,

$$\rho_f|_{G_p} \sim \begin{pmatrix} \psi \chi_p^{k-1} \delta^{-1} & * \\ 0 & \delta \end{pmatrix} \tag{2}$$

where G_p is the decomposition group at p determined by the embedding ι_p and δ is a non-trivial unramified character.

Now suppose f has complex multiplication. Then, there is an imaginary quadratic field $F/\mathbf{Q}$, an algebraic Hecke character $\widehat{\eta}$ of F of type $(k-1, 0)$ and a (continuous) character $\eta : G_F \to \overline{\mathbf{Q}}_p^\times$, corresponding to $\widehat{\eta}$ under class field theory (and the choices of ι_∞ and ι_p), such that $\rho_f = \mathrm{Ind}_F^{\mathbf{Q}} \eta$. Letting τ denote an element in $G_{\mathbf{Q}}$ not in G_F, we thus have

$$\rho_f(g) = \begin{cases} \begin{pmatrix} \eta(g) & \\ & \eta(\tau g \tau^{-1}) \end{pmatrix} & \text{if } g \in G_F, \\[2em] \begin{pmatrix} & \eta(g \tau^{-1}) \\ \eta(\tau g) & \end{pmatrix} & \text{if } g \notin G_F. \end{cases} \tag{3}$$

By (2), this implies that $G_p \subseteq G_F$, i.e. p splits in F. Let $\theta_F : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ be the quadratic character associated to the extension $F/\mathbf{Q}$ by class field theory (so that $\theta_F(g) = -1$ if, and only if, $g \notin G_F$). Then, $\theta_F(p) = 1$ and since complex conjugation acts non-trivially on F, $\theta_F(\mathrm{Frob}_\infty) = -1$, i.e. θ_F is odd.

For the rest of this section, we will consider even $n = 2m$. This is a necessary condition for a trivial zero to occur; see the proof of Theorem A for a discussion of the case of odd n. Taking the nth symmetric power of ρ_f yields

$$\mathrm{Sym}^n \rho_f(g) = \begin{cases} \begin{pmatrix} \eta^n(g) & & & & \\ & \eta^{n-1}(g)\eta(\tau g \tau^{-1}) & & & \\ & & \ddots & & \\ & & & & \eta^n(\tau g \tau^{-1}) \end{pmatrix} & \text{if } g \in G_F, \\[3em] \begin{pmatrix} & & & \eta^n(g \tau^{-1}) \\ & & \cdot^{\cdot^{\cdot}} & \\ & \eta^{n-1}(\tau g)\eta(g\tau^{-1}) & & \\ \eta^n(\tau g) & & & \end{pmatrix} & \text{if } g \notin G_F \end{cases} \tag{4}$$

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[3]In the sense that $\mathrm{ord}_p(\iota_p \iota_\infty^{-1} a_p) = 0$, where a_p is the pth Fourier coefficient.

in some basis, say, $v_0, \ldots, v_n$. The entry in the middle of this matrix representation is given by

$$\begin{cases} \eta^m(g)\eta^m(\tau g\tau^{-1}) & \text{if } g \in G_F, \\ \eta^m(\tau g)\eta^m(g\tau^{-1}) & \text{if } g \notin G_F. \end{cases}$$

We may rewrite this as $(\theta_F \det \rho_f)^m$. To identify the remaining entries, we change basis to $v_0, v_n, v_1, v_{n-1}, v_2, \ldots, v_{n/2}$. The matrix is then the direct sum of $(\theta_F \det \rho_F)^m$ and two-by-two blocks that we can write as

$$\begin{cases} \begin{pmatrix} \eta^{n-2i}(g)(\theta_F(g) \det \rho_f(g))^i & \\ & \eta^{n-2i}(\tau g\tau^{-1})(\theta_F(g) \det \rho_f(g))^i \end{pmatrix} & \text{if } g \in G_F, \\[2ex] \begin{pmatrix} & \eta^{n-2i}(g\tau^{-1})(\theta_F(g) \det \rho_f(g))^i \\ \eta^{n-2i}(\tau g)(\theta_F(g) \det \rho_f(g))^i & \end{pmatrix} & \text{if } g \notin G_F, \end{cases}$$

for $0 \le i \le m - 1$. As alluded to in the introduction, we are interested in generalizing results obtained for the L-invariant of $\mathrm{ad}^0\rho_f$, the trace zero endomorphisms of ρ_f. In the representation theory of $\mathrm{GL}(2)$,

$$\mathrm{ad}^0 \mathrm{Std} \cong (\mathrm{Sym}^2\mathrm{Std}) \otimes \det^{-1},$$

where Std denotes the standard two-dimensional representation of $\mathrm{GL}(2)$. We thus view

$$\rho_n := (\mathrm{Sym}^n\rho_f) \otimes (\det \rho_f)^{-m} \tag{5}$$

as the proper generalization of the adjoint representation. We remark that the associated motive has weight 0 so that the centre of the functional equation is $s = 1/2$ and it will turn out that the trivial zeroes will occur at the near-central points. Twisting the above matrices by $(\det \rho_f)^{-m}$, we obtain

$$\rho_n \cong \theta_F^m \oplus \bigoplus_{i=0}^{m-1} \left(\left(\mathrm{Ind}_F^{\mathbf{Q}} \eta^{2(m-i)} \right) \otimes \chi_p^{(m-i)(1-k)} \otimes \psi^{i-m}\theta_F^i \right). \tag{6}$$

Since $\widehat{\eta}$ is of type $(k-1, 0)$, the Artin reciprocity map applied to $\eta^{2(m-i)}$ gives us Hecke characters of F of type $(2(m-i)(k-1), 0)$ that we denote $\widehat{\eta}_j$ (for $1 \le j := m - i \le m$). These in turn yield newforms f_j' of weight $2j(k-1) + 1$, which we must twist by $\psi^{-j}\theta_F^i$. This latter operation may result in imprimitive forms, but we let f_j denote the unique newform (of level $N_j | N\mathfrak{f}_{\psi^{-j}\theta_F^i}^2$) whose Hecke eigenvalues agree with those of the possibly imprimitive form $f_j' \otimes \psi^{-j}\theta_F^i$ outside of the primes dividing $N\mathfrak{f}_{\psi^{-j}\theta_F^i}$. By (6), we get a product decomposition of the L-function of ρ_n as

$$L(s, \rho_n) = L(s, \theta_F^m) \cdot \prod_{j=1}^{m} L(s + j(k-1), f_j). \tag{7}$$

The critical integers $C_{n,k}$ of $L(s, \rho_n)$ depend on the parities of both m and k. When m is odd and

k is even, the critical integers are

$$C_{n,k} = \{-(k-1)+1, -(k-1)+3, \ldots, -2, 0, 1, 3, 5, \ldots, k-3, k-1\}. \tag{8}$$

If m and k are both odd, then

$$C_{n,k} = \{-(k-1), -(k-1)+2, \ldots, -2, 0, 1, 3, 5, \ldots, k-4, k-2\}. \tag{9}$$

Otherwise, $C_{n,k}$ consists of the positive even integers $\leq k-1$ and the negative odd integers $\geq -k$. These are simple to work out from the Hodge structures of the corresponding motives.

In the next section, we will p-adically interpolate the values of $L(s, \rho_n)$ at these integers by taking the product of the p-adic L-functions attached to each L-function on the right-hand side of (7). This is a sensible approach since the critical integers of ρ_n are contained in the intersection of the critical integers of the L-functions on the right-hand side of the decomposition. Indeed, the critical integers of a cusp form of weight k' are $\{1, 2, \ldots, k'-1\}$, those of an odd Dirichlet character (such as $\theta_F^m = \theta_F$ for m odd) are the positive odd and non-positive even integers, and those of an even Dirichlet character (such as $\theta_F^m = \mathbf{1}$ for m even) are the positive even and negative odd integers.

2 The analytic L-invariant

As is appropriate, we refer the reader to the section "Notation and conventions" for our notation and conventions on p-adic L-functions. We remark that we will only consider the p-adic L-function of ρ_n corresponding to the ordinary refinement.

Suppose that f is a p-ordinary newform of weight $k \geq 2$, level $\Gamma_1(N)$, with $p \nmid N$, and Nebentypus character ψ. By the ordinarity assumption, (exactly) one of the roots of the Hecke polynomial $x^2 - a_p x + \psi(p)p^{k-1}$ of f is a p-adic unit; we denote it by α_p and denote the non-unit root by β_p. Then, through the work of Manin, Amice–Vélu, Višik, and Shimura, there exist $\Omega_f^+, \Omega_f^- \in \mathbf{C}^\times$ and a $\mathbf{C}_p$-valued p-adic measure μ_f on $\Gamma^\times$ such that for any finite order character $\chi \in \mathfrak{X}^\times$ and $1 \leq a \leq k-1$, an integer,

$$\iota_p^{-1} \int_{\Gamma^\times} \chi \chi_p^a d\mu_f = \iota_\infty^{-1} \left(\frac{L(a, f, \chi^{-1})}{\Omega_{a,f,\chi}} \right) \left(1 - \frac{p^{a-1}\chi(p)}{\alpha_p} \right) \left(1 - p^{-a}\chi^{-1}(p)\beta_p \right), \tag{10}$$

where, writing $\mathfrak{f}_\chi = p^r$,

$$\frac{1}{\Omega_{a,f,\chi}} = \frac{\Gamma(a)\mathfrak{f}_{\chi^{-1}}^a}{\tau(\chi^{-1})(-2\pi i)^a \alpha_p^r} \cdot \begin{cases} \Omega_f^+ & \text{if } (-1)^a\chi(-1) = 1 \\ \Omega_f^- & \text{if } (-1)^a\chi(-1) = -1. \end{cases}$$

As discussed in [MTT86, §II.15], no trivial zeroes can occur for f as above (namely, p-ordinary with $p \nmid N$).

Consequently, the trivial zeroes of ρ_n must come from the Dirichlet character θ_F^m. This character is trivial unless m is odd. Let us study the trivial case first. It follows from the classical results of

Kubota–Leopoldt and Iwasawa that there is a $\mathbf{C}_p$-valued p-adic pseudo-measure[4] $\mu_{\mathbf{1}}$ on $\Gamma^\times$ such that for any finite order character $\chi \in \mathfrak{X}^\times$ and for an integer a

$$\iota_p^{-1} \int_{\Gamma^\times} \chi \chi_p^a d\mu_{\mathbf{1}} = \iota_\infty^{-1} \left(\frac{L(a, \chi^{-1})}{\Omega_{a,\mathbf{1},\chi}} \right) \cdot \begin{cases} \left(1 - p^{-a}\chi^{-1}(p)\right) & \text{if } (-1)^a \chi(-1) = -1, a \leq 0 \\ \left(1 - p^{a-1}\chi(p)\right) & \text{if } (-1)^a \chi(-1) = 1, a \geq 1, \end{cases} \tag{11}$$

where

$$\frac{1}{\Omega_{a,\theta,\chi}} = \begin{cases} 1 & \text{if } (-1)^a \chi(-1) = -1, a \leq 0, \\ \frac{2\Gamma(a)\mathfrak{f}_{\chi^{-1}}^a}{\tau(\chi^{-1})(-2\pi i)^a} & \text{if } (-1)^a \chi(-1) = 1, a \geq 1. \end{cases}$$

Thus, no trivial zero can occur for the trivial character $\mathbf{1}$. Indeed, any trivial zero of this function must occur at $a = 0$ or 1. If $a = 0$, then, for criticality, χ must be odd, and in particular non-trivial, so that $\chi^{-1}(p) = 0$. The same reasoning applies to $a = 1$.

Now, let θ be an odd Dirichlet character of conductor prime to p. Then, there is a $\mathbf{C}_p$-valued p-adic measure[5] μ_θ on $\Gamma^\times$ such that for any finite order character $\chi \in \mathfrak{X}^\times$ and for an integer a

$$\iota_p^{-1} \int_{\Gamma^\times} \chi \chi_p^a d\mu_\theta = \iota_\infty^{-1} \left(\frac{L(a, \theta\chi^{-1})}{\Omega_{a,\theta,\chi}} \right) \cdot \begin{cases} \left(1 - p^{-a}(\theta\chi^{-1})(p)\right) & \text{if } (-1)^a \chi(-1) = 1, a \leq 0, \\ \left(1 - p^{a-1}(\theta^{-1}\chi)(p)\right) & \text{if } (-1)^a \chi(-1) = -1, a \geq 1, \end{cases} \tag{12}$$

where

$$\frac{1}{\Omega_{a,\theta,\chi}} = \begin{cases} 1 & \text{if } (-1)^a \chi(-1) = 1, a \leq 0, \\ \frac{2\Gamma(a)\mathfrak{f}_{\theta\chi^{-1}}^a}{\tau(\theta\chi^{-1})(-2\pi i)^a} & \text{if } (-1)^a \chi(-1) = -1, a \geq 1. \end{cases}$$

Here, the parities work out so that trivial zeroes can occur. Again, a trivial zero can only occur for $a = 0$ or 1, in which cases one must also have that χ is trivial and $\theta(p) = 1$.

Inspired by the decomposition of L-functions in (7), we define a ($\mathbf{C}_p$-valued) p-adic measure (or pseudo-measure if m is even) μ_{ρ_n} on $\Gamma^\times$ as a convolution

$$\mu_{\rho_n} := \mu_{\theta_F^m} * \underset{j=1}{\overset{m}{\text{\Large$*$}}} \left(\chi_p^{j(k-1)} \mu_{f_j} \right). \tag{13}$$

As we have seen that no trivial zero can occur when m is even, we restrict from now on to the interesting case where m is odd. It then follows from (12) and (10) that for any finite order character

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[4] Let us remark that we have normalized our choices so that

$$L_{p,i}(s, \mathbf{1}) = \begin{cases} L_p^{\mathrm{KL}}(s, \omega^{1-i}) & \text{if } i \text{ is odd,} \\ L_p^{\mathrm{KL}}(1-s, \omega^i) & \text{if } i \text{ is even,} \end{cases}$$

where $L_p^{\mathrm{KL}}(s, \chi)$ is the classical Kubota–Leopoldt p-adic L-function attached to the even character χ.

[5] For odd θ, our choice is such that

$$L_{p,i}(s, \theta) = \begin{cases} L_p^{\mathrm{KL}}(s, \theta\omega^{1-i}) & \text{if } i \text{ is even,} \\ L_p^{\mathrm{KL}}(1-s, \theta^{-1}\omega^i) & \text{if } i \text{ is odd.} \end{cases}$$

$\chi \in \mathfrak{X}^{\times}$ and any $a \in C_{n,k}$

$$\iota_p^{-1} \int_{\Gamma^{\times}} \chi \chi_p^a d\mu_{\rho_n} = \iota_{\infty}^{-1} \left(\frac{L(a, \theta_F \chi^{-1})}{\Omega_{a,\theta_F,\chi}} \cdot \prod_{j=1}^{m} \frac{L(a + j(k-1), f_j, \chi^{-1})}{\Omega_{a+j(k-1),f_j,\chi}} \right)$$

$$\times \prod_{j=1}^{m} \left(1 - \frac{p^{a+j(k-1)-1}\chi(p)}{\alpha_{j,p}} \right) \left(1 - p^{-a-j(k-1)}\chi^{-1}(p)\beta_{j,p} \right)$$

$$\times \begin{cases} (1 - p^{-a}(\theta_F \chi^{-1})(p)) & \text{if } (-1)^a \chi(-1) = 1, a \leq 0 \\ (1 - p^{a-1}(\theta_F \chi)(p)) & \text{if } (-1)^a \chi(-1) = -1, a \geq 1, \end{cases}$$

$$= \iota_{\infty}^{-1} \left(\frac{L(a, \rho_n, \chi^{-1})}{\Omega_{a,\rho_n,\chi}} \right) \times \prod_{j=1}^{m} \left(1 - \frac{p^{a+j(k-1)-1}\chi(p)}{\alpha_{j,p}} \right) \left(1 - p^{-a-j(k-1)}\chi^{-1}(p)\beta_{j,p} \right)$$

$$\times \begin{cases} (1 - p^{-a}(\theta_F \chi^{-1})(p)) & \text{if } (-1)^a \chi(-1) = 1, a \leq 0 \\ (1 - p^{a-1}(\theta_F \chi)(p)) & \text{if } (-1)^a \chi(-1) = -1, a \geq 1, \end{cases} \tag{14}$$

where $\alpha_{j,p}$ (resp. $\beta_{j,p}$) is the unit root (resp. non-unit root) of the Hecke polynomial of f_j and we have set

$$\Omega_{a,\rho_n,\chi} := \Omega_{a,\theta_F,\chi} \cdot \prod_{j=1}^{m} \Omega_{a+j(k-1),f_j,\chi}.$$

The trivial zeroes of ρ_n occur at $L_p(0, \rho_n, \mathbf{1}) = L_{p,0}(0, \rho_n)$ and $L_p(1, \rho_n, \mathbf{1}) = L_{p,1}(1, \rho_n)$, i.e. in the 0th and 1st branch, respectively. The definition (13) gives product decompositions

$$L_{p,i}(s, \rho_n) = L_{p,i}(s, \theta_F) \cdot \prod_{j=1}^{m} L_{p,i}(s + j(k-1), f_j). \tag{15}$$

For each j, $s = j(k-1)$ and $1 + j(k-1)$ are the near-central points for f_j. The work of Jacquet–Shalika in [JSh76] shows that the archimedean L-values at these points are non-zero. We may thus conclude that

$$L_{p,i}(j(k-1), f_j) \neq 0 \neq L_{p,i}(1 + j(k-1), f_j).$$

Therefore, taking the derivative of both sides in (15), we obtain for $i = 0, 1$

$$L'_{p,i}(i, \rho_n) = L'_{p,i}(i, \theta_F) \cdot \prod_{j=1}^{m} L_{p,i}(i + j(k-1), f_j). \tag{16}$$

This reduces the problem of computing the L-invariants of ρ_n to those of θ_F. These values were obtained by Ferrero and Greenberg in [FeG78], Gross and Koblitz in [GrKo79], and placed within a theoretical framework by Gross in [Gr81]. For an odd Dirichlet character θ, let $R_p(\theta)$ be the p-adic regulator Gross defines in equation (2.10) of his article. Let h_F denote the class number of F and factor $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ into prime ideals of F, where $\mathfrak{p}$ corresponds to the embedding ι_p. Let $\overline{\pi}$ be a generator of the principal ideal $\overline{\mathfrak{p}}^{h_F}$ (well-defined up to a root of unity). We may now state the first theorem of our paper.

Theorem A. Let p be an odd prime and let f be a p-ordinary newform of weight $k \geq 2$, Γ_1-level prime to p, Nebentypus character ψ, and CM by F. Let ρ_f denote the p-adic Galois representation attached to f. If n is a positive integer and $m = \lfloor n/2 \rfloor$, then the p-adic L-function of $\rho_n = (\mathrm{Sym}^n \rho_f) \otimes (\det \rho_f)^{-m}$ has a trivial zero if, and only if, $n = 2m$ with m odd. In such a case, the trivial zeroes of ρ_n are of order 1 and occur at the near-central points $L_p(0, \rho_n, \mathbf{1})$ and $L_p(1, \rho_n, \mathbf{1})$. The analytic L-invariants are given by

$$\mathcal{L}_p^{\mathrm{an}}(1, \rho_n, \mathbf{1}) = -\frac{2 \log_p(\overline{\pi})}{h_F} \tag{17}$$

and

$$\mathcal{L}_p^{\mathrm{an}}(0, \rho_n, \mathbf{1}) = -\mathcal{L}_p^{\mathrm{an}}(1, \rho_n, \mathbf{1}) \tag{18}$$

and occur in the following interpolation formulas for $i = 0$ and 1

$$L'_{p,i}(i, \rho_n) = \mathcal{L}_p^{\mathrm{an}}(i, \rho_n, \mathbf{1}) \mathcal{E}^+(i, \rho_n, \mathbf{1}) \frac{L(i, \rho_n)}{\Omega_{i, \rho_n, \mathbf{1}}} \tag{19}$$

where

$$\mathcal{E}^+(i, \rho_n, \mathbf{1}) = \prod_{j=1}^m \left(1 - \frac{p^{i+j(k-1)-1}\chi(p)}{\alpha_{j,p}} \right) \left(1 - p^{-i-j(k-1)}\chi^{-1}(p)\beta_{j,p} \right).$$

We also have the identities

$$\mathcal{L}_p^{\mathrm{an}}(1, \rho_n, \mathbf{1}) = \mathcal{L}_p^{\mathrm{an}}(1, \theta_F, \mathbf{1}) = R_p(\theta_F) = \mathcal{L}_p^{\mathrm{Gr}}(1, \theta_F, \mathbf{1}). \tag{20}$$

Remark 2.1. Before justifying the rest of this theorem, we make a few remarks.

(i) This theorem solves part (i) of conjecture 0.1. Part (ii) follows from the three equalities

$$\mathcal{L}_p^{\mathrm{Gr}}(1, \theta_F, \mathbf{1}) = \mathcal{L}_p^{\mathrm{Gr}}(1, \rho_n, \mathbf{1}), \tag{21}$$

$$\mathcal{L}_p^{\mathrm{Gr}}(1, \theta_F, \mathbf{1}) = \mathcal{L}_p^{\mathrm{Gr}}(1, \rho_n^\vee, \mathbf{1}), \tag{22}$$

and

$$\mathcal{L}_p^{\mathrm{Gr}}(0, \rho_n, \mathbf{1}) = -\mathcal{L}_p^{\mathrm{Gr}}(1, \rho_n^\vee, \mathbf{1}) \tag{23}$$

where $\rho_n^\vee$ is the linear dual of ρ_n. See §3 for the proof of these equalities and an explanation of our notation for Greenberg's L-invariant.

(ii) This shows that the L-invariant of ρ_n only depends on the CM field F, in particular it is independent of n. Greenberg has raised the question of whether the L-invariant is independent of n for all newforms f. In [G94, p. 170], he shows that this is true from his L-invariant when either E is an elliptic curve with complex multiplication or with split, multiplicative reduction at p. The question for general f is still open.

(iii) In Theorem B, we will show that $\mathcal{L}_p^{\mathrm{an}}(1, \rho_n, \mathbf{1}) = -2 \log_p(\alpha_p)/(k-1)$, thus generalizing the result of Dabrowski and Delbourgo in [DabDel97, Theorem 3.6.1].

(iv) The L-invariant of θ_F can be written in terms of Morita's p-adic Gamma function, see [FeG78, Proposition 1].

(v) Using the results of Gross–Koblitz in [GrKo79], these L-invariants can be expressed in terms of Gauss sums.

Proof. We have already explained everything in this theorem except for the order of the trivial zeroes, the values of the L-invariants, and the case of odd n. We begin with n even. That the analytic L-invariants of ρ_n and θ_F are equal follows from (16). As indicated in footnotes above, we have

$$L_{p,1}(s, \theta_F) = L_p^{\mathrm{KL}}(1 - s, \theta_F \omega) \tag{24}$$

and

$$L_{p,0}(s, \theta_F) = L_p^{\mathrm{KL}}(s, \theta_F \omega). \tag{25}$$

That these Kubota–Leopoldt p-adic L-functions vanish to order exactly 1 at $s = 0$ is Proposition 2 of [FeG78].

Note that

$$L_{p,0}(s, \theta_F) = L_{p,1}(1 - s, \theta_F),$$

so

$$L'_{p,0}(0, \theta_F) = -L'_{p,1}(1, \theta_F).$$

Combining this with the functional equation for Dirichlet L-functions gives (18) and reduces the proof of the case $i = 0$ to that of $i = 1$: indeed, if

$$L'_{p,1}(1, \theta_F) = \mathcal{L}_p^{\mathrm{an}}(1, \theta_F, \mathbf{1}) \frac{L(1, \theta_F)}{\Omega_{1, \theta_F, \mathbf{1}}},$$

then the functional equation says that the fraction on the right is simply $L(0, \theta_F)$.

The equality between Gross' p-adic regulator and Greenberg's L-invariant is Proposition 5 of [G94]. $L_p^{\mathrm{KL}}(1 - s, \theta_F \omega)$ is what Gross denotes (except for the superscript that we have added) by $L_p^{\mathrm{Gross}}(\omega \otimes \theta_F, 1 - s)$. His conjecture 2.12 is that

$$(L_p^{\mathrm{Gross}})'(\omega \otimes \theta_F, 0) = R_p(\theta_F) A(\theta_F),$$

where we see from [Gr81, (2.16)] that $A(\theta_F) = -L(0, \theta_F)$. This conjecture is proved in §4 of his paper for the situation in which we find ourselves. Using the functional equation as above yields

$$(L_p^{\mathrm{Gross}})'(\omega \otimes \theta_F, 0) = -R_p(\theta_F) \frac{L(1, \theta_F)}{\Omega_{1, \theta_F, \mathbf{1}}}.$$

Since

$$L'_{p,1}(1, \theta_F) = -(L_p^{\mathrm{Gross}})'(\omega \otimes \theta_F, 0)$$

we have proved (20).

In [FeG78, §4] and [GrKo79, (4.12)], one finds

$$(L_p^{\mathrm{KL}})'(0, \theta_F \omega) = \frac{4}{w_F} \log_p(\overline{\pi}),$$

where w_F is the number of roots of unity in F. By the analytic class number formula,

$$L(0, \theta_F) = \frac{2h_F}{w_F},$$

so

$$(L_p^{\mathrm{KL}})'(0, \theta_F \omega) = \frac{2\log_p(\overline{\pi})}{h_F} L(0, \theta_F),$$

from which the formula in (17) follows.

It remains to deal with the case of odd n. Briefly, the L-function decomposes into a product of L-functions of modular forms (i.e. no Dirichlet character shows up in the decomposition). All of the modular forms that show up will once again be ordinary at p and have level prime to p. So, as mentioned above, none of them will have trivial zeroes. □

3   The arithmetic L-invariant

We will now briefly explain how the results in Theorem A are those predicted by Greenberg's theory of trivial zeroes and we will use [H-PhD, Theorem A] to prove another formula for the L-invariant. As a reference for Greenberg's theory see [H10, §1] or [G94]. We will follow the notation of the former.

We begin by remarking that Greenberg's theory predicts when a Galois representation V has a trivial zero at $s = 1$. So, the L-invariant denoted by $\mathcal{L}(V)$ in [H10] is what we denote here by $\mathcal{L}_p^{\mathrm{Gr}}(1, V, \mathbf{1})$. Furthermore, saying that a trivial zero of V should occur at $s = a_0$ is the same as saying that it should occur at $s = 1$ for $V(1 - a_0)$.

The discussion in [H10, §1.3] on the expected location of trivial zeroes of the symmetric powers of f applies here, as well.[6] Therefore, we can say that the trivial zeroes we found on the analytic side in the previous section are exactly those predicted by Greenberg's arithmetic theory and they occur with the expected order. It then remains to check that the L-invariants are as predicted. Let ρ_n be defined as in the previous section, with $n = 2m$ and m odd. By definition, $\mathcal{L}_p^{\mathrm{Gr}}(0, \rho_n, \mathbf{1}) = \mathcal{L}(\rho_n(1))$ which in turn is defined[7] to be $-\mathcal{L}((\rho_n(1))^\vee(1))$. Since $(\rho_n(1))^\vee(1) = \rho_n^\vee$, this gives equation (23) mentioned above. There is a condition that must be satisfied in order for Greenberg's definition of the L-invariant of V to make sense, namely that the balanced Selmer group $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$ vanishes. As in Proposition 1.3 of [H10], we have that $\overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_n) = H_f^1(\mathbf{Q}, \rho_n)$ where the latter is the likely more familiar Bloch–Kato Selmer group defined in [BK90]. Similarly for $\rho_n^\vee$. Using some deep results of Rubin's on the main conjecture for imaginary quadratic fields, we obtain the following result.

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[6] There the Nebentypus was assumed to be trivial, whereas here we only assume it has conductor prime to p. However, the same arguments go through without modification.

[7] This definition is implicit in [G94].

Proposition 3.1. *The balanced Selmer groups $\overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_n)$ and $\overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_n^\vee)$ vanish.*

Proof. The decomposition in (6) gives that

$$\overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_n) = H_f^1(\mathbf{Q}, \rho_n) = H_f^1(\mathbf{Q}, \theta_F) \oplus \bigoplus_{j=1}^m H_f^1(\mathbf{Q}, \rho_{f_j}(j(k-1))).$$

The vanishing of $H_f^1(\mathbf{Q}, \theta_F)$ is a classical result that can be reduced to the finiteness of the class number. For $H_f^1(\mathbf{Q}, \rho_{f_j}(j(k-1)))$, the vanishing is a deep result. Kato explains in [Ka04, §15] how to deduce it from Rubin's results on the main conjecture for imaginary quadratic fields. Note that we are looking at Selmer groups of modular forms at non-central points. The Selmer groups at the central point lie deeper and are the subject of the Birch and Swinnerton-Dyer conjecture.

For $\overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_n^\vee)$, the vanishing follows as above. Indeed, $\rho_n^\vee$ decomposes into the Dirichlet character θ_F and twists of newforms $f_j^\vee$ which are contragredient to the f_j. □

Next, we state a lemma whose proof we omit.

Lemma 3.2. *If $V = \bigoplus_{j=0}^m V_j$ is a decomposition of V as $G_{\mathbf{Q}}$-modules, then*

$$\mathcal{L}_p^{\mathrm{Gr}}(1, V, \mathbf{1}) = \prod \mathcal{L}_p^{\mathrm{Gr}}(1, V_j, \mathbf{1})$$

where the product is over only those V_j which are expected to have a trivial zero.

Applying this to the decomposition of ρ_n with $V_0 = \theta_F$ and $V_j = \rho_{f_j}(j(1-k))$, for $j = 1, \ldots, m$, we get that

$$\mathcal{L}_p^{\mathrm{Gr}}(1, \rho_n, \mathbf{1}) = \mathcal{L}_p^{\mathrm{Gr}}(1, \theta_F, \mathbf{1}),$$

since it is easy to verify that, for $j \geq 1$, V_j are not expected to have trivial zeroes. This equality is the one mentioned in (21). The same reasoning shows that

$$\mathcal{L}_p^{\mathrm{Gr}}(1, \rho_n^\vee, \mathbf{1}) = \mathcal{L}_p^{\mathrm{Gr}}(1, \theta_F, \mathbf{1}),$$

which is equation (22) above. Let us now state and prove the second, and final, theorem of our paper.

Theorem B. *With the hypotheses of Theorem A, we have, for $i = 0$ or 1,*

$$\mathcal{L}_p^{\mathrm{an}}(i, \rho_n, \mathbf{1}) = \mathcal{L}_p^{\mathrm{Gr}}(i, \rho_n, \mathbf{1}) \tag{26}$$

and

$$\mathcal{L}_p^{\mathrm{Gr}}(1, \rho_n, \mathbf{1}) = -\frac{2\log_p(\alpha_p)}{k-1}. \tag{27}$$

Proof. The equality of the analytic L-invariant with Greenberg's L-invariant follows from equations (21), (22), and (23) proved above.

To prove the formula for Greenberg's L-invariant, we compute the $n = 2$ case using the author's result in [H-PhD, Theorem A]. Indeed, we have shown in Theorem A that the L-invariant is

independent of n, so we are free to pick an n. For $n = 2$, Theorem A of [H-PhD] states that in the current situation

$$\mathcal{L}_p^{\mathrm{Gr}}(1, \rho_2, \mathbf{1}) = -\frac{2\alpha_p'}{\alpha_p}$$

and we will now explain the notation α_p'. We will require some results of Hida theory and we use [Hi11] as a reference. Given a p-ordinary newform f of level prime to p, Hida theory provides a p-adic analytic family $\mathcal{F}$ of p-adic modular forms deforming the p-stabilization f_p of f, where

$$f_p(z) := f(z) - \beta_p f(pz).$$

The p-stabilization has the effect that the pth Fourier coefficient of f_p is α_p. We may write $\mathcal{F}$ as a q-expansion

$$\mathcal{F} = \sum_{n \geq 1} a_n(s) q^n$$

where $a_n(s)$ is a p-adic analytic function of s on some neighbourhood of k and $a_n(k)$ is the nth Fourier coefficient of f_p. In particular, $a_p(k) = \alpha_p$. Then α_p' is defined as

$$\alpha_p' := \left. \frac{da_p(s)}{ds} \right|_{s=k}.$$

Alternatively, $\mathcal{F}$ provides a Galois deformation of ρ_f and $a_p(s)$ is the p-adic analytic function that interpolates the unit roots of Frobenius at p in this deformation.

We are thus trying to show that the logarithmic derivative of $a_p(s)$ at $s = k$ is given by $\log_p(a_p(k))/(k-1)$. This will be true if $a_p(s)$ is an exponential function (to the correct base), which it is. This is because the Hida family of f interpolates the inductions of the Hecke characters η^r with varying r (see, for example, [Hi86, §7]). Explicitly, it is shown in [Hi11, p. 1337] that

$$a_p(s) = \zeta \exp_p\left((s-1)\frac{\log_p(\overline{\pi})}{h_F \log_p(1+p)} \log_p(1+p)\right) = \exp_p\left((s-1)\frac{\log_p(\overline{\pi})}{h_F}\right) \tag{28}$$

where ζ is a root of unity; indeed, this formula is written there as

$$\zeta t^{\log_p(\overline{\pi})/(h_F \log_p(1+p))}$$

where[8] $t = 1 + T \in \mathbb{I}$. Moreover, a p-adic analytic function is obtained from this latter expression by sending t to $(1+p)^{s-1}$. Since the p-adic logarithm of a root of unity is zero, taking the logarithm on both sides of (28) yields

$$\log_p(a_p(s)) = (s-1)\frac{\log_p(\overline{\pi})}{h_F}, \tag{29}$$

so that the logarithmic derivative of $a_p(s)$ at $s = k$ is $\log_p(\overline{\pi})/h_F$. Plugging $s = k$ into (29) gives

$$\log_p(\alpha_p) = \log_p(a_p(k)) = (k-1)\frac{\log_p(\overline{\pi})}{h_F}.$$

---

[8]Here, $\mathbb{I}$ is the irreducible component of Hida's ordinary p-adic Hecke algebra corresponding to f_p.

Therefore,

$$\mathcal{L}_p^{\mathrm{Gr}}(1, \rho_n, \mathbf{1}) = -\frac{2\alpha_p'}{\alpha_p} = -\frac{2\log_p(\alpha_p)}{k-1},$$

as desired. $\square$

Remark 3.3.

(i) This formula for the L-invariant generalizes that in [DabDel97, Theorem 3.6.1]. There, Dabrowski and Delbourgo are considering the case where $n = 2$ and f corresponds to an elliptic curve over $\mathbf{Q}$ with CM. We would like to thank Antonio Lei for bringing this result to our attention.

(ii) If one could prove that, for at least one n, the analytic L-invariant equals Greenberg's L-invariant independently of the work of Ferrero–Greenberg and Gross–Koblitz, then the proof above would provide a new proof of their formulas for $(L_p^{\mathrm{KL}})'(0, \theta_F\omega)$.

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